An Introduction to Resolvent kernels
for Dirichlet BVPs on Finite Networks

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We aim here at introducing the basic terminology and results on *the Resolvent Kernels* **associated with the Dirichlet BVP on finite networks. Firstly, we define the discrete analogue of the Green and Poisson Kernels and we established its main properties and relation. Then, we define the Dirichlet to Neumann map and study its relation with the Poisson and Green kernels.**

1. PRELIMINARES

Throughout the paper, $\Gamma = (V, E)$ denotes a simple, finite and connected graph without loops, with vertex set V and edge set E. Two different vertices, $x, y \in V$, are called *adjacent*, which will be represented by $x \sim y$, if $\{x, y\} \in E$.

Given a vertex subset $F \subset V$, we denote by F^c its complementary in V and we call *boundary* and *closure of* F, the sets $\delta(F) = \{x \in V : x \sim y \text{ for some } y \in F\}$ and $\bar{F} = F \cup \delta(F)$, respectively. If $F \subset V$ is a proper subset, we say that F is *connected* if for any $x, y \in V$ there exists a path joined x and y whose vertices are all in F. It is easy to prove that $\bar{F}$ is connected when F is.

The sets of functions and non-negative functions on V are denoted by $\mathcal{C}(V)$ and $\mathcal{C}^+(V)$ respectively. If $u \in \mathcal{C}(V)$, its *support* is given by $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. Moreover, if $x \in V$, we denote by ε_y the Dirac function; that is, $\varepsilon_y(x) = 0$ if $x \neq y$ and $\varepsilon_y(y) = 1$. If F is a non empty subset of V, its characteristic function is denoted by χ_F and we can consider the sets $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$ and $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$. We call *weight on* F any function $\sigma \in \mathcal{C}^+(F)$ such that $\mathsf{supp}(\sigma) = F$. The set of weights on F is denoted by $\Omega(F)$.

We call *conductance on* Γ a function $c: V \times V \longrightarrow \mathbb{R}^+$ such that $c(x, y) > 0$ iff $x \sim y$. We call *weighted network* any triple (Γ, c, ν), where c is a conductance on Γ and $\nu \in \Omega(V)$. In what follows we consider fixed the network (Γ, c, ν) and we refer to it simply by Γ. The function $\kappa \in \mathcal{C}^+(V)$ defined as $\kappa(x) = \int_V c(x, y)\, dy$ for any $x \in V$ is called the *degree of* Γ that clearly satisfies that $\mathsf{supp}(\kappa) = V$. In addition, for any proper subset $F \subset V$ we call *the boundary degree of* F the function $\kappa_F \in \mathcal{C}\big(\delta(F)\big)$ defined as $\kappa_F(x) = \int_F c(x, y)\, dy$ for any $x \in \delta(F)$, that clearly satisfies that $\mathsf{supp}(\kappa_F) = \delta(F)$.

The *combinatorial Laplacian* or simply the *Laplacian* of Γ is the linear operator $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$(1) \qquad \mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_V c(x, y) \big(u(x) - u(y)\big)\, dy, \quad x \in V.$$

Given $q \in \mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathcal{L}_q : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$.

If F is a proper subset of V, for each $u \in \mathcal{C}(\bar{F})$ we define the *normal derivative of u on F* as the function in $\mathcal{C}(\delta(F))$ given by

$$(2) \qquad \left(\frac{\partial u}{\partial \mathsf{n}_F} \right)(x) = \frac{1}{\nu(x)} \int_F c(x,y) \big(u(x) - u(y) \big) \, dy, \quad \text{for any } x \in \delta(F).$$

The *normal derivative on F* is the operator $\dfrac{\partial}{\partial \mathsf{n}_F} : \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\delta(F))$ that to any $u \in \mathcal{C}(\bar{F})$ assigns its normal derivative on F.

The relation between the values of the Schrödinger operator with potential q on F and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*.

$$\int_F v \, \mathcal{L}_q(u) \, d\nu = \frac{1}{2} \int_{\bar{F}} \int_{\bar{F}} c_F(x,y)(u(x) - u(y))(v(x) - v(y)) \, dx dy + \int_F quv \, d\nu - \int_{\delta(F)} v \, \frac{\partial u}{\partial \mathsf{n}_F} \, d\nu,$$

where $u, v \in \mathcal{C}(\bar{F})$ and $c_F = c \cdot \chi_{(\bar{F} \times \bar{F}) \setminus (\delta(F) \times \delta(F))}$. A direct consequence of the above identity is the so–called *Second Green Identity*

$$\int_F \left(v \, \mathcal{L}_q(u) - u \, \mathcal{L}_q(v) \right) d\nu = \int_{\delta(F)} \left(u \, \frac{\partial v}{\partial \mathsf{n}_F} - v \, \frac{\partial u}{\partial \mathsf{n}_F} \right) d\nu, \quad \text{for all } u, v \in \mathcal{C}(\bar{F}).$$

In the sequel we consider a connected proper subset $F \subset V$. Given functions $q \in \mathcal{C}(F)$, $f \in \mathcal{C}(F)$, $h \in \mathcal{C}(\delta(F))$ a *Dirichlet boundary value problem on F* consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$(3) \qquad \mathcal{L}_q(u) = f \quad \text{on} \quad F \quad \text{and} \quad u = h \quad \text{on} \quad \delta(F).$$

The quadratic form associated with the boundary value problem (3) is the function $\mathcal{Q} \colon \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by

$$(4) \qquad \mathcal{Q}(u) = \frac{1}{2} \int_{\bar{F} \times \bar{F}} c_F(x,y) \big(u(x) - u(y) \big)^2 dx dy + \int_F q \, u^2 d\nu.$$

Next we obtain a sufficient condition guaranteeing that $\mathcal{Q}$ is positive definite. To do this we consider $\sigma \in \Omega(\bar{F})$ and the following identity for each $u \in \mathcal{C}(V)$ and any $x, y \in \bar{F}$

$$(5) \qquad \big(u(x) - u(y) \big)^2 = \sigma(x)\sigma(y) \left(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)} \right)^2 + \frac{u^2(x)}{\sigma(x)} \big(\sigma(x) - \sigma(y) \big) - \frac{u^2(y)}{\sigma(y)} \big(\sigma(x) - \sigma(y) \big).$$

Using the above identity, usually called *Doob transform*, we get that for each $u \in \mathcal{C}(V)$

$$(6) \qquad \mathcal{Q}(u) = \frac{1}{2} \int_{\bar{F} \times \bar{F}} c_F(x,y) \, \sigma(x)\sigma(y) \left(\frac{u(x)}{\sigma(x)} - \frac{u(y)}{\sigma(y)} \right)^2 dx dy + \int_F (q - q_\sigma) \, u^2 d\nu + \int_{\delta(F)} \frac{1}{\sigma} \frac{\partial \sigma}{\partial \mathsf{n}_F} u^2 \, d\nu.$$

where $q_\sigma(x) = -\sigma^{-1} \mathcal{L}(\sigma)(x)$ for all $x \in F$.

Proposition 1.1 ([2], Proposition 4). *Given $q \in \mathcal{C}(F)$, $\mathcal{Q}$ is positive definite on $\mathcal{C}(F)$ if there exists a weight $\sigma \in \Omega(\bar{F})$ such that $q \geq q_\sigma$ on F. If in addition $\dfrac{\partial \sigma}{\partial \mathsf{n}_F} \geq 0$ on $\delta(F)$, then $\mathcal{Q}$ is positive semi–definite on $\mathcal{C}(\bar{F})$ and positive definite when either $q \neq q_\sigma$ on F or $\dfrac{\partial \sigma}{\partial \mathsf{n}_F} \neq 0$ on $\delta(F)$.*

From now on we will suppose that there exists a weight $\sigma \in \Omega(\bar{F})$ such that $q \geq q_\sigma$ on F. This hypothesis implies that Problem (3) has a unique solution for any data.

2. MONOTONICITY AND CONDENSER PRINCIPLE

Our aim in this section is to establish the monotonicity property of the Schrödinger operators that was proved in [2, Proposition 4.10].

Proposition 2.1 (Monotonicity). *If $u \in \mathcal{C}(\bar{F})$ verifies that $\mathcal{L}_q(u) \geq 0$ on F and $u \geq 0$ on $\delta(F)$, then either $u > 0$ on F or $u = 0$ on $\bar{F}$.*

Proof. Let $v = \sigma^{-1}u$ and $x \in F$ such that $v(x) = \min_{z \in F}\{v(z)\}$. If $v(x) > 0$, then $u > 0$ on F. So, suppose that $v(x) \leq 0$. Then, $v(x) \leq v(y)$ for all $y \in \bar{F}$, since $v \geq 0$ on $\delta(F)$, and therefore,

$$0 \leq \mathcal{L}_q(u)(x) = \frac{1}{\nu(x)} \int_{\bar{F}} c(x,y)\sigma(y)\big(v(x) - v(y)\big)dy + (q(x) - q_\sigma(x))\sigma(x)v(x) \leq 0,$$

which implies that $v(x) = v(y)$ for all $y \in \bar{F}$, since $\bar{F}$ is connected. Hence $u = a\sigma$, with $a \in \mathbb{R}^+$, but $u(x) \leq 0$, then $u = 0$.

If we denote $v = \sigma^{-1}u$, we first prove that $v \in \mathcal{C}^+(\bar{F})$. Indeed, if $x \in F$ is such that $v(x) = \min_{z \in F}\{v(z)\}$ it suffices to prove that $v(x) \geq 0$, or equivalently that if $v(x) \leq 0$ then necessarily $v(x) = 0$. Suppose that $v(x) \leq 0$. Then $v(x) \leq v(y)$ for all $y \in \bar{F}$ and therefore,

$$0 \leq \mathcal{L}_q(u)(x) = \frac{1}{\nu(x)} \int_{\bar{F}} c(x,y)\sigma(y)\big(v(x) - v(y)\big)dy + (q(x) - q_\sigma(x))\sigma(x)v(x) \leq 0,$$

which implies that $v(x) = v(y)$ for all $y \in \bar{F}$ such that $x \sim y$ since $\bar{F}$ is connected. Hence $u = a\sigma$, with $a \geq 0$, but $u(x) \leq 0$, then $u = 0$. $\square$

Let us introduce the well-known *Condenser Principle*. Suppose that $\delta(F) = A \cup B$, where A, B are not empty sets and $A \cap B = \emptyset$. Then, $\bar{F}$ is called *condenser* with *positive* and *negative plates* A and B, respectively, when F is connected with a medium of conductance q. When $q = 0$ on F, we say that F is *isolated* of the surrounding media.

The *Condenser Problem* consists in the following boundary value problem

(7) $$\mathcal{L}_q(u) = 0 \quad \text{on} \quad F, \quad u = \sigma \quad \text{on } A \text{ and} \quad u = 0 \text{ on } B.$$

Of course, the above definition has sense, since Problem (7) has a unique non-null solution.

The following result follows from the application of monotonicity property to the solution of the Condenser Problem.

Proposition 2.2 (Condenser Principle). *If $u \in \mathcal{C}(\bar{F})$ is the unique solution of the Condenser Problem (7), then $0 < u < \sigma$ on F. Moreover, it is satisfied that $\dfrac{\partial u}{\partial \mathsf{n}_F} > \dfrac{\partial \sigma}{\partial \mathsf{n}_F}$ on A, $\dfrac{\partial u}{\partial \mathsf{n}_F} < 0$ on B.*

Proof. The positiveness of u follows directly from Proposition 2.1, since $u = \sigma > 0$ on A. Moreover, if $v = \sigma - u$ then $\mathcal{L}_q(v) = (q - q_\sigma)\sigma \geq 0$ on F, $v = 0$ on A and $v = \sigma$ on B. Therefore, applying again Proposition 2.1, $v > 0$ on F. Finally, if $x \in A$ and $y \in B$ we get that $u(x) = \sigma(x)$, $u(y) = 0$ and moreover

$$\frac{\partial u}{\partial \mathsf{n}_F}(x) = \frac{1}{\nu(x)} \int_F c(x,z)\big(\sigma(x) - u(z)\big)\,dz = \frac{\partial \sigma}{\partial \mathsf{n}_F}(x) + \frac{1}{\nu(x)} \int_F c(x,z)\big(\sigma(z) - u(z)\big)\,dz$$

$$\frac{\partial u}{\partial \mathsf{n}_F}(y) = \frac{1}{\nu(y)} \int_F c(y,z)\big(u(y) - u(z)\big)\,dz = -\frac{1}{\nu(y)} \int_F c(y,z)u(z)\,dz.$$

The last claims follow bearing in mind that $\displaystyle\int_F c(x,z)\big(\sigma(z) - u(z)\big)\,dz > 0$ and $\displaystyle\int_F c(y,z)u(z)\,dz > 0$. $\square$

3. Resolvent kernels for the Dirichlet Problem on Finite Networks

Suppose that $F \subset V$ is a proper subset and we consider for $f \in \mathcal{C}(F)$, the Dirichlet problem

$$(8) \qquad \qquad \mathcal{L}_q u = f \ \text{ on } \ F \ \text{ and } \ u = 0 \ \text{ on } \ \delta(F).$$

The above problem has a unique solution that can be expressed by means of its Green kernel. A function $G : \bar{F} \times F \longrightarrow \mathbb{R}$ is called *the Green kernel of the BVP (8)* iff for all $y \in F$, the function $G_y = G(\cdot, y)$ is the unique solution of

$$(9) \qquad \qquad \mathcal{L}_q(G_y) = \varepsilon_y \ \text{ on } \ F \ \text{ and } \ G_y = 0 \ \text{ on } \ \delta(F).$$

Proposition 3.1. *The Green kernel verifies the following properties:*

   (i) $\nu(x)G(x,y) = \nu(y)G(y,x)$, *for all* $x, y \in F$.
   (ii) *For any* $x, y \in F$, $G(x,y) > 0$ *and* $\sigma(y)G(x,y) \le \sigma(x)G(y,y)$ *for any* $x \ne y$.
   (iii) *For any* $f \in \mathcal{C}(F)$, *the unique solution of Problem (8) is given by*

$$u(x) = \int_F G(x,y)f(y)\,dy = \sum_{y \in F} G(x,y)f(y), \ \text{ for any } \ x \in \bar{F}.$$

Proof. (i) For any $x, y \in F$, let $u = G(\cdot, y)$ and $v = G(\cdot, x)$, then $u = v = 0$ on $\delta(F)$ and from the Second Green Identity we have that

$$\nu(x)G(x,y) = \nu(x)u(x) = \int_F u\varepsilon_x d\nu = \int_F u\mathcal{L}_q v \, d\nu = \int_F v\mathcal{L}_q u \, d\nu = \int_F v\varepsilon_y d\nu = \nu(y)v(y) = \nu(y)G(y,x).$$

(ii) Let $y \in F$ and consider $u = G_y$. Then, from the monotonicity property, $u > 0$ on F, since $\mathcal{L}_q(u) \ge 0$ on F and non null.

On the other hand, if H denotes any connected component of $F \setminus \{y\}$, then $y \in \delta(H)$ and moreover $\delta(H) \setminus \{y\} \subset \delta(F)$.

If $\delta(H) \setminus \{y\} \ne \emptyset$, we consider $v = \dfrac{\sigma(y)}{G(y,y)}\, G_y$. Therefore,

$$\mathcal{L}_q(v) = 0 \ \text{ on } \ H, \ \ v(y) = \sigma(y) \ \text{ and } \ v = 0 \ \text{ on } \ \delta(H) \setminus \{y\}$$

and applying the Condenser Principle we get that $0 < v < \sigma$ on H, and the result follows.

If $\delta(H) = \{y\}$, we consider $v = \sigma - \dfrac{\sigma(y)}{G(y,y)}\, G_y$. Therefore, $\mathcal{L}_q(v) = (q - q_\sigma)\sigma$ on H and $v(y) = 0$. If $q \ne q_\sigma$ on H, applying the monotonicity, we obtain that $v > 0$ on H and the result follows. If $q = q_\sigma$ on H, then $v = 0$, since the Dirichlet Problem has uniqueness of solutions; that is, $\sigma = \dfrac{\sigma(y)}{G(y,y)}\, G_y$ on H.

(iii) Let $x \in F$, then

$$\mathcal{L}_q(u)(x) = \int_F \mathcal{L}_q(G_y)(x)f(y)\,dy = \int_F \varepsilon_y(x)f(y)\,dy = f(x).$$

Moreover, for any $x \in \delta(F)$, $u(x) = \displaystyle\int_F G(x,y)f(y)\,dy = 0$, since $G_y(x) = 0$. Therefore, u is the unique solution of Problem (8). $\qquad \square$

REMARK: From the proof of the part (ii) of the above proposition, it follows that for any $y \in F$, the inequality $\sigma(y)G(x,y) \le \sigma(x)G(y,y)$ for any $x \ne y$ is strict, except when $\delta(H) = \{y\}$ and $q = q_\sigma$ on H.

For $h \in \mathcal{C}(\delta(F))$ consider now the Dirichlet problem

$$(10) \qquad \qquad \mathcal{L}_q u = 0 \ \text{ on } \ F \ \text{ and } \ u = h \ \text{ on } \ \delta(F).$$

The above problem has a unique solution that can be expressed by means of its Poisson kernel. A function $P : \bar{F} \times \delta(F) \longrightarrow \mathbb{R}$ is called *the Poisson kernel of the BVP (10)* iff for all $y \in \delta(F)$, the function $P_y = P(\cdot, y)$ is the unique solution of

$$(11) \qquad \qquad \mathcal{L}_q(P_y) = 0 \ \text{ on } \ F, \ \ P_y(y) = 1 \ \text{ and } \ P_y = 0 \ \text{ on } \ \delta(F) \setminus \{y\}.$$

Proposition 3.2. *The Poisson kernel verifies the following properties:*

(i) $\nu(x)\dfrac{\partial P_y}{\partial \mathsf{n}_F}(x) = \nu(y)\dfrac{\partial P_x}{\partial \mathsf{n}_F}(y)$, *for all* $x, y \in \delta(F)$.

(ii) *For any* $y \in \delta(F)$, $0 < \sigma(y)P(x,y) \leq \sigma(x)$ *for any* $x \in F$ *and* $\dfrac{\partial P_y}{\partial \mathsf{n}_F}(x) < 0$ *for any* $x \in \delta(F)$, $x \neq y$.

(iii) *For any* $h \in \mathcal{C}(\delta(F))$, *the unique solution of Problem (10) is given by*

$$u(x) = \int_{\delta(F)} P(x,y)h(y)\, dy, \quad \text{for any } x \in \bar{F}.$$

Proof. (i) If for any $x, y \in \delta(F)$, we consider $u = P_y$ and $v = P_x$, then $\mathcal{L}_q(u) = \mathcal{L}_q(v) = 0$ on F and from the Second Green Identity we have that

$$\nu(x)\frac{\partial P_y}{\partial \mathsf{n}_F}(x) = \nu(x)\frac{\partial u}{\partial \mathsf{n}_F}(x) = \int_{\delta(F)}\frac{\partial u}{\partial \mathsf{n}_F}P_x d\nu = \int_{\delta(F)}\frac{\partial v}{\partial \mathsf{n}_F}P_y d\nu = \nu(y)\frac{\partial v}{\partial \mathsf{n}_F}(y) = \nu(y)\frac{\partial P_x}{\partial \mathsf{n}_F}(y).$$

(ii) Let $y \in \delta(F)$ and consider $u = P_y$. Then, from the monotonicity property, $u > 0$ on F, since $u(y) > 0$.
On the other hand, if $\delta(F) \setminus \{y\} \neq \emptyset$, we consider $v = \sigma(y)\, P_y$. Therefore,

$$\mathcal{L}_q(v) = 0 \quad \text{on} \quad F, \quad v(y) = \sigma(y) \quad \text{and} \quad v = 0 \quad \text{on} \quad \delta(F) \setminus \{y\}$$

and applying the Condenser Principle we get that $0 < v < \sigma$ on F, $\dfrac{\partial v}{\partial \mathsf{n}_F}(x) < 0$ for any $x \in \delta(F)$ such that $x \neq y$ and the results follow.

If $\delta(F) = \{y\}$, we consider $v = \sigma - \sigma(y)\, P_y$. Therefore, $\mathcal{L}_q(v) = (q - q_\sigma)\sigma$ on F and $v(y) = 0$. If $q \neq q_\sigma$ on F, applying the monotonicity, we obtain that $v > 0$ on F and the result follows. If $q = q_\sigma$ on F, then $v = 0$, since the Dirichlet Problem has uniqueness of solutions; that is, $\sigma = \sigma(y)\, P_y$ on F.

(iii) Let $x \in F$, then

$$\mathcal{L}_q(u)(x) = \int_{\delta(F)} \mathcal{L}_q(P_y)(x)h(y)\, dy = 0.$$

Moreover, for any $x \in \delta(F)$, $u(x) = \displaystyle\int_{\delta(F)} P(x,y)h(y)\, dy = h(x)$, since $P_y = \varepsilon_y$. Therefore, u is the unique solution of Problem (10). $\qquad\square$

REMARK: From the proof of the part (ii) of the above proposition, it follows that for any $y \in \delta(F)$, the inequality $\sigma(y)P(x,y) \leq \sigma(x)$ for any $x \in F$ is strict, except when $\delta(F) = \{y\}$ and $q = q_\sigma$ on F.

Given $f \in \delta(F)$, we know that u is the solution of Problem (10) iff $u = v + h$ where v is the unique solution of Problem (8) with data $f = -\mathcal{L}(h)$. This equivalence leads us to obtain a relation between Green and Poisson kernels.

Proposition 3.3. *For any* $x \in \bar{F}$ *and any* $y \in \delta(F)$ *the following identity holds:*

$$P(x,y) = \varepsilon_y(x) - \frac{\nu(y)}{\nu(x)}\frac{\partial G_x}{\partial \mathsf{n}_F}(y).$$

Proof. Let $y \in \delta(F)$ and consider $u = P_y$. Then $u = \varepsilon_y + v$, where v is the unique solution of $\mathcal{L}_q(v) = -\mathcal{L}(\varepsilon_y)$ on F and $v = 0$ on $\delta(F)$. From part (iii) of Proposition (9) and applying the Second Green Identity, we get that if $x \in F$,

$$v(x) = -\int_F G(x,z)\mathcal{L}(\varepsilon_y)(z)\, dz = -\frac{1}{\nu(x)}\int_F G_x(z)\mathcal{L}(\varepsilon_y)(z)\, d\nu(z) = -\frac{1}{\nu(x)}\int_F \mathcal{L}(G_x)(z)\varepsilon_y(z)\, d\nu(z)$$

$$+ \frac{1}{\nu(x)}\int_{\delta(F)}\frac{\partial \varepsilon_y}{\partial \mathsf{n}_F}(z)G_x(z)\, d\nu(z) - \frac{1}{\nu(x)}\int_{\delta(F)}\frac{\partial G_x}{\partial \mathsf{n}_F}(z)\varepsilon_y(z)\, d\nu(z) = -\frac{\nu(y)}{\nu(x)}\frac{\partial G_x}{\partial \mathsf{n}_F}(y). \qquad\square$$

4. The Kernel of the Dirichlet to Neumann Map

The map $\Lambda \colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any $h \in \mathcal{C}(\delta(F))$, the function $\Lambda(h) = \dfrac{\partial u_h}{\partial \mathsf{n}_F}$, where u_h is the unique solution of Problem (10) with data h, is called *Dirichlet-to-Neumann map*. The following result is a straightforward consequence of the Green Identities.

Proposition 4.1. *The Dirichlet-to-Neumann map is a self–adjoint operator and its associated quadratic form is given by*

$$\int_{\delta(F)} h\Lambda(h)d\nu = \mathcal{Q}(u_h).$$

In particular, if $\dfrac{\partial \sigma}{\partial \mathsf{n}_F} \geq 0$ on $\delta(F)$, then the Dirichlet-to-Neumann map is positive semi–definite and positive definite when either $q \neq q_\sigma$ on F or $\dfrac{\partial \sigma}{\partial \mathsf{n}_F} \neq 0$ on $\delta(F)$.

Proof. Given $f, h \in \mathcal{C}(\delta(F))$, by applying the Second Green Identity we obtain that

$$\int_{\delta(F)} f\Lambda(h)\, d\nu = \int_{\delta(F)} f\frac{\partial u_h}{\partial \mathsf{n}_F}\, d\nu = \int_{\delta(F)} h\frac{\partial u_f}{\partial \mathsf{n}_F}\, d\nu = \int_{\delta(F)} h\Lambda(f)\, d\nu$$

and hence Λ is a self–adjoint operator. On the other hand, from the First Green Identity we get

$$\int_{\delta(F)} h\Lambda(h)\, d\nu = \int_{\delta(F)} u_h \frac{\partial u_h}{\partial \mathsf{n}_F}\, d\nu = \frac{1}{2}\int_{\bar{F}\times\bar{F}} c_F(x,y)\big(u_h(x) - u_h(y)\big)^2 dxdy + \int_F qu_h^2\, d\nu = \mathcal{Q}(u_h)$$

and the last claim follows from Proposition 1.1.   □

Proposition 4.2. *Given $h \in \mathcal{C}(\delta(F))$, then for any $x \in \delta(F)$ we get that*

$$\Lambda(h)(x) = \int_{\delta(F)} \frac{\partial P_y}{\partial \mathsf{n}_F}(x)h(y)\, dy;$$

that is, the kernel of the Dirichlet-to-Neumann is $K(x,y) = \dfrac{\partial P_y}{\partial \mathsf{n}_F}(x)$. In particular $K(x,y) < 0$ for $x, y \in \delta(F)$ such that $x \neq y$. In addition, if $\dfrac{\partial \sigma}{\partial \mathsf{n}_F} \geq 0$ on $\delta(F)$, then $K(y,y) > 0$ for any $y \in \delta(F)$.

Proof. From Proposition 3.2, we get that for any $x \in \bar{F}$, $u_h(x) = \displaystyle\int_{\delta(F)} P(z,y)h(y)\, dy$ and hence if $x \in \delta(F)$, then

$$\Lambda(h)(x) = \frac{\partial u_h}{\partial \mathsf{n}_F}(x) = \int_{\delta(F)} \frac{\partial P_y}{\partial \mathsf{n}_F}(x)\, h(y)\, dy. \qquad □$$

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